

## **A Logic for Reasoning about Persuasion**

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**Abstract.** In the article, we introduce a sound and complete deductive system ( $\mathcal{AG}_n$ ) which can be used to reason about persuasion process performed in distributed systems of agents in circumstances of uncertain and incomplete information. In order to express degrees of beliefs of these agents, we adopt methods of Logic of Graded Modalities. To represent degrees' changes resulting from the persuasion, we apply tools of Algorithmic Logic and Dynamic Logic. As a result, we interpret arguments as actions which lead to change of grades of agents' beliefs.

### **1. Introduction**

The aim of the paper is to propose the logic which allows to reason about persuasion process performed in distributed systems with uncertain and incomplete information. The name of the formalism is Multimodal Logic of Actions and Graded Beliefs  $\mathcal{AG}_n$ . In the article we show that  $\mathcal{AG}_n$  is sound and complete.

The body of the article consists of three chapters. In the section “The System  $\mathcal{AG}_n$ ” we show the deductive system for reasoning about persuasion. We present the syntax and semantics of its language

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as well as a set of corresponding axioms and inference rules. In the next chapter, we prove soundness and completeness of  $\mathcal{AG}_n$ . And finally - in the section “Examples of expressiveness” we show what can be expressed in the  $\mathcal{AG}_n$  language with respect to the persuasion process. Still, before we move to those issues we want to start with (1) giving motivation for our interest in the topic of persuasion, (2) discussing the adequacy of the logic  $\mathcal{AG}_n$  for investigating the process of convincing and (3) briefly presenting other approaches to the subject we examine.

### 1.1. Motivation

First of all, we want to give motivations for introducing the logic  $\mathcal{AG}_n$  discussing the reasons for which we are concerned in the persuasion process. Convincing is an important topic in the research on artificial as well as human societies. Its significance results in a great deal of interest and a large number of publications in philosophy, logic or theoretical computer science (see e.g. [8, 17, 21, 23, 26]).

Once the complex societies (artificial or human) are studied, a researcher encounters the issue of distributed systems with available data which is incomplete and uncertain. This means that: (i) information is distributed among individuals (some data is not known by every agent, but could be collected from particular entities), (ii) not every data is available (e.g. an access to some sources of information - like sensors - is unreachable), (iii) data is unreliable (e.g. a sensor can be broken). Clearly, in such circumstances the conflicts easily arise. From the point of view of the society, **resolving the conflict** is a top necessity since disagreements unable agents to cooperate. The persuasion may then be used as a tool for reaching a consensus and creating common attitude between individuals.

We understand **persuasion** as an action initiated by the conflict and aimed to influence beliefs. The logic  $\mathcal{AG}_n$  allows to reason about the effects that convincing brings about and to evaluate particular cases of persuasion - what chances for success has the persuader in the specific situation, how strong and difficult the victory would be, etc. To conclude, we are interested in the persuasion process as in the powerful tool of the system of agents which allows to resolve conflicts between them.<sup>1</sup>

### 1.2. Adequacy of description

The next question becomes: whether the language of  $\mathcal{AG}_n$  is adequate to describe the persuasion. To give an answer, we have to know what characteristics constitute this process. Let us discuss it first.

Once we aim to describe the persuasion executed in distributed systems with incomplete and uncertain information, we must be able to express belief-attitudes in more nuanced way than “yes-or-no” framework. Imagine a situation on an airplane when a captain has to decide what maneuver to perform to avoid a danger. Say that an officer tries to convince the captain they should turn off some engines. How often will it be the case that the captain is absolutely sure of his decision? In order to represent such types of persuasion, we assign to beliefs various **degrees of uncertainty**. As a result, we are able to describe not only “black-and-white” types of convincing (i.e. before: *I did not believe the thesis* after: *I do*, or the opposite way), but also such types of persuasion that increase the grade of certainty in not a full range (e.g. after the persuasion *I do believe the thesis stronger, but not absolutely*). Notice that what we are directly interested in here is not just modeling gradation of beliefs, but rather **changes of these degrees** throughout the process of persuasion (cf. [5, 6]).

<sup>1</sup>We may think about the persuasion as about a tool for resolving conflicts by agents themselves, i.e. without the interference of a user of the system.

Secondly, a persuasion can be executed by three **parties of conflict**: proponent (a party that proposes a thesis and defends it with arguments), opponent (a party that opposes a thesis and possibly attacks it with counterarguments) and audience (a party that evaluates the arguments of both sides and chooses the winner) [4].

Finally, since we focus on exploring the success in convincing (which ensures the resolution of the conflict), the persuasion should be understood as a dynamic phenomenon. Clearly the success is impossible to achieve unless the proponent performs some action - say, show or does something. That is, if an argument exists only in the persuader's mind, there is no chance to influence audience beliefs. However, once the proponent performs the argument, there is a possibility (although no guarantee) that he will succeed. Thus, we want **arguments to be actions** - the persuader gives an argument in a sense that he executes some action.

	<b>Characteristics</b>	<b>Formalism</b>
<b>Object of persuasion</b>	graded beliefs	graded doxastic logic
<b>Subject of persuasion</b>	parties of contention	multimodal logic
<b>Tools of persuasion</b>	arguments	logic of action

Table 1. The nature of persuasion and its formal description.

To adequately describe distinguished characteristics of persuasion, we choose specific formal methods (see Table 1). Firstly, in order to represent degrees of beliefs we use **graded doxastic system** inspired by Logic of Graded Modalities of Wiebe van der Hoek and John Jules Meyer [15, 18]. Their formalism is useful for those applications in which agents have to reason and work on the basis of data laden with exceptions (that is, when it is necessary to act despite of using unreliable sources of information). It helps us to capture various grades of belief-attitudes of the agents in conflict. However, we modify their logic to make it more appropriate for our needs. We change their epistemic approach into the doxastic one, since persuasion refers to beliefs not knowledge. Furthermore, we introduce additional belief-operator which allows to express the degrees of agent's uncertainty in a more direct way.

Secondly, since individuals play different roles in a conflict (of proponent, opponent and audience), we use methods of **multimodal logic** [18]. Thus, there are so many operators of a given type as many individuals in a society. For a system of  $n$  individuals, an agent with the number  $i$  (where  $i \in \{1, \dots, n\}$ ) has assigned the belief-operator with the number  $i$ . On the semantic level, the model is extended by  $n$  doxastic accessibility relations - one for each agent.

Finally, we interpret arguments in the **logic of actions** which is inspired by elements of Algorithmic Logic [19] and Dynamic Logic [14]. Yet we modify their notation so that we can indicate who is a performer of an action (who is a proponent of persuasion). Moreover, we are able to express who is an addressee of argument (who is an audience) by pointing out whose beliefs are about to change under the influence of arguments. On the semantic level, we understand persuasion as an action which adds or eliminates transitions in an audience's doxastic accessibility relation (changing thereby audience's opinions about the reality). Such a representation enables us to research the issue of how arguments influence persuaded agent's beliefs.

### 1.3. Related work

This is not the first paper to consider the change of agent’s cognitive attitudes. We give a brief overview of the related works here. *Dynamic epistemic logic* models change of agent’s knowledge combining epistemic modal logic with dynamic modal logic (see e.g. [20, 3, 2, 9]).<sup>2</sup> *Dynamic doxastic logic* describes change of beliefs by means of doxastic modal logic and dynamic modal logic (e.g. [25, 16]). In the *belief revision* formalism, a set of agent’s beliefs is changed due to expansion, contraction or revision in light of new (possibly conflicting) evidence (e.g. [1, 13, 24]). *BDI-logics* are combinations of logics on the BDI-approach including, but not limited to epistemic, doxastic or dynamic ones (e.g. [8, 27]).<sup>3</sup>

Despite relatively much work on providing logic for reasoning about change of agent’s cognitive attitudes, there are some fundamental differences between those approaches and ours. The *contribution* of the presented paper is that we do not focus on the change of beliefs (knowledge), but on the **change of the degrees** of beliefs represented in the manner of the Logic of Graded Modalities. The second difference arises from the fact that the issue of convincing was not the main interest in these other formalisms.<sup>4</sup> We want our logic not to be applied to reason about change of cognitive attitudes but rather to **reason about persuasion**. Indeed, we show that the logic of actions and graded beliefs is a highly expressive and useful tool for studying the persuasion process undertaken by agents.

## 2. The System $\mathcal{AG}_n$

In this paragraph we show a deductive system which we use for the persuasion theory. It is the Multi-modal Logic of Actions and Graded Beliefs ( $\mathcal{AG}_n$ ). The logic we consider is an extension of a propositional language in which there are propositional variables, program variables, and apart from the usual propositional connectives there is one program connective. Moreover, we add some modalities for expressing properties concerning beliefs as well as actions. In order to model degrees of beliefs we use Logic of Graded Modalities (LGM) [15, 18] while the part concerning arguments is inspired by logics of programs like Algorithmic Logic (AL) [19] and Dynamic Logic (DL) [14]. We do not use the whole apparatus offered by DL or AL which are much more rich formalizations than the one we explore. For our reasons it is sufficient to use only basic elements of these logics.

### 2.1. Syntax and semantics

Let  $V_0$  denote an at most enumerable set of *propositional variables* (also called *propositions*)  $p, r, s, \dots$  and  $\Pi_0$  an at most enumerable set of *program variables* (also called *atomic actions*)  $a_1, a_2, \dots$ . Propositional variables represent atomic assertions such as: “the temperature equals 10 degrees”, “the ther-

<sup>2</sup>The proposals of dynamic epistemic logic were inspired by the public announcement logic of Plaza [22] (with the exception of the Moore’s proposal [20]). The Plaza’s logic became modified by generalizing dynamic operators for truthful public announcement to more complex epistemic actions.

<sup>3</sup>Observe that BDI-approach concentrates rather on representing beliefs, desires and intentions than change of cognitive states of agents.

<sup>4</sup>The process of convincing was considered in BDI framework in [8]. However, even though F. Dignum, B. Dunin-Keplicz and R. Verbrugge talk about actions and beliefs in the context of persuasion, it is still other approach than we have in mind. The key difference is that they understand actions as specific speech acts performed during agents’ dialogue, while we want actions to be executed arguments.

“thermometer is wrongly placed” etc. which can be either true or false. Further, program variables represent things happening. In our formalism, they express giving arguments - both verbal (like uttering words e.g. saying “you should move the thermometer”) and nonverbal (like moving the thermometer).

In addition, we assume the boolean connectives:  $\neg$  (negation, “not”),  $\wedge$  (conjunction, “and”),  $\vee$  (disjunction, “or”),  $\rightarrow$  (implication, “if ... then ...”),  $\leftrightarrow$  (equivalence, “if and only if”, “iff”) and one program connective:  $;$  which is a sequential composition operator. By means of sequential compositions we compose *schemes of programs* which are defined as finite sequences of atomic **actions**:  $a_1; a_2; \dots; a_k$ . Intuitively, the program  $a_1; a_2$  for  $a_1, a_2 \in \Pi_0$  means “Do  $a_1$ , then do  $a_2$ ”.

The set of well-formed schemes of programs  $\Pi$  is defined as follows:

- $a \in \Pi$  for any  $a \in \Pi_0$ ,
- if  $P_1, P_2 \in \Pi$ , then  $P_1; P_2 \in \Pi$ .

There are considered many program connectives in logics of programs, e.g. nondeterministic choices or iteration operations. However, sequential compositions are sufficient for our needs.

The last components of the language are modalities. We use modality  $M$  for reasoning about beliefs shared by agents in persuasion and modalities  $\diamond$  and  $\square$  for reasoning about actions (arguments) they perform. The intended interpretation of  $M_i^d \alpha$  is that there are **more** than  $d$  states which are considered by an agent  $i$  and verify  $\alpha$ . Whereas, formulas  $\diamond(i : P)\alpha$  and  $\square(i : P)\alpha$  say that after execution of a program  $P$  by an agent  $i$  a condition  $\alpha$  may or must be true, respectively. This means that if  $P = (a_1; \dots; a_k)$  for  $a_1, \dots, a_k \in \Pi_0$  and the formula  $\diamond(i : P)\alpha$  is valid then **it is possible** that after giving arguments  $a_1; \dots; a_k$ , the thesis  $\alpha$  holds. On the contrary, if the formula  $\square(i : P)\alpha$  is valid then **always** after giving arguments  $a_1; \dots; a_k$ , the thesis  $\alpha$  holds.

Now, we can define the set  $F$  of all well-formed expressions of  $\mathcal{AG}_n$ . A grammar of the language is written in Backus-Naur Form (BNF) as follows:

$$\alpha ::= p \mid \neg \alpha \mid \alpha \vee \alpha \mid M_i^d \alpha \mid \diamond(i : P)\alpha,$$

where  $p$  is a propositional variable,  $d$  is a natural number,  $P$  is a program scheme,  $i$  is a name of an agent. For simplicity, for names of agents we apply natural numbers, so we assume that  $i \in \{1, \dots, n\}$  for some natural  $n$ .

Other boolean connectives are defined from  $\neg$  and  $\vee$  in the standard way. The necessity operator  $\square$  is the modal dual of the possibility operator  $\diamond$  and is defined as  $\square(i : P)\alpha \leftrightarrow \neg \diamond(i : P)\neg \alpha$ . We use  $B_i^d \alpha$  as an abbreviation for  $\neg M_i^d \neg \alpha$  - at most  $d$  states considered by  $i$  refute  $\alpha$ . We use also  $M_i!^d \alpha$  where  $M_i!^0 \alpha \leftrightarrow B_i^0 \neg \alpha$ ,  $M_i!^d \alpha \leftrightarrow M_i^{d-1} \alpha \wedge \neg M_i^d \alpha$ , if  $d > 0$ . From the definition above, it is clear that  $M_i!^d$  means “exactly  $d$ ”. The most important formula that we shall use in reasoning about a persuasion process is  $M_i!^{d_1, d_2} \alpha$  which is an abbreviation for  $M_i!^{d_1} \alpha \wedge M_i!^{d_2} true$ . It should be read as “ $i$  believes  $\alpha$  with a degree  $\frac{d_1}{d_2}$ ”. Thereby, by a **degree of beliefs** of agents we mean the ratio of  $d_1$  to  $d_2$ , i.e. the ratio of the number of states which are considered by an agent  $i$  and verify  $\alpha$  to the number of all states which are considered by this agent. It is easy to observe that  $0 \leq \frac{d_1}{d_2} \leq 1$ . Intuitively, if an agent believes a thesis  $\alpha$  with a degree 1 then he is absolutely sure that  $\alpha$  holds while if he believes  $\alpha$  with a degree 0 then he is absolutely certain  $\alpha$  is false.

The semantics of the language is based on the notions of *valuation* and *interpretation*. A valuation is a function which assigns a logical value “false” (denoted by  $\mathbf{0}$ ) or “true” (denoted by  $\mathbf{1}$ ) to every

propositional variable. An interpretation assigns to every program variable and every agent a binary relation in a non-empty set of states  $S$ . Every state will be understood to be an abstraction of a concrete situation on which the behaviour of the program and the value of any formula depends. Every state carries information about the valuations of propositional variables. Furthermore, we consider a doxastic function which assign to every agent a binary relation which will give interpretation of the believe operator.

**Definition 2.1.** Let  $Agt = \{1, 2, \dots, n\}$  be a finite set of names of agents. By a semantic **model** we mean a Kripke structure  $\mathcal{M} = (S, RB, I, v)$  where

- $S$  is a non-empty set of states (the universe of the structure),
- $RB$  is a doxastic function,  $RB : Agt \longrightarrow 2^{S \times S}$ , where for every  $i \in Agt$ , the relation  $RB(i)$  is serial, transitive and Euclidean,
- $I$  is an interpretation of the program variables,  $I : \Pi_0 \longrightarrow (Agt \longrightarrow 2^{S \times S})$ , where for every  $a \in \Pi_0$  and  $i \in Agt$ , the relation  $I(a)(i)$  is serial, and  $I(Id)(i) = \{(s, s) : s \in S\}$ , where  $Id$  is a program constant which means identity,
- $v$  is a function which assigns to every state a valuation of propositional variables  $v : S \longrightarrow \{0, 1\}^{V_0}$  and for every  $s \in S$ ,  $v(s)(true) = 1$ , where  $true$  is a propositional constant.

Notice that we assume that for every  $i \in Agt$  the relation  $RB(i)$  is serial, transitive and Euclidean. Furthermore, we do not require this relation to be reflexive since we want the operator  $M$  to model beliefs rather than knowledge of individuals. In standard epistemic logic, it is assumed that an individual cannot know facts that are not true, so reflexivity is desirable.

Function  $I$  can be extended in a simple way to define interpretation of any program scheme. Let  $I_\Pi : \Pi \longrightarrow (Agt \longrightarrow 2^{S \times S})$  be a function defined by mutual induction on the structure of  $P \in \Pi$  as follows:

- $I_\Pi(a)(i) = I(a)(i)$  for  $a \in \Pi_0$  and  $i \in Agt$ ,
- $I_\Pi(P_1; P_2)(i) = I_\Pi(P_1)(i) \circ I_\Pi(P_2)(i) = \{(s, s') \in S \times S : \exists s'' \in S ((s, s'') \in I_\Pi(P_1)(i) \text{ and } (s'', s') \in I_\Pi(P_2)(i))\}$  for  $P_1, P_2 \in \Pi$  and  $i \in Agt$ .

In other words,  $(s, s') \in I_\Pi(P)(i)$  for  $P = (a_1; \dots; a_k)$  and  $i \in Agt$  iff there exists a sequence of states  $s_0, \dots, s_k$  such that  $(s_{j-1}, s_j) \in I(a_j)(i)$  for  $j = 1, \dots, k$ . Intuitively, it means that the state  $s'$  can be achieved from the state  $s$  if the agent  $i$  performs actions  $a_1, \dots, a_k$  in order they appear.

Now, we are ready to define **semantics** of formulas of  $\mathcal{AG}_n$ .

**Definition 2.2.** For a given structure  $\mathcal{M} = (S, RB, I, v)$  and a given state  $s \in S$  the boolean value of the formula  $\alpha$  is denoted by  $\mathcal{M}, s \models \alpha$  and is defined inductively as follows:

$$\begin{aligned}
\mathcal{M}, s \models p & \quad \text{iff} \quad v(s)(p) = 1, \text{ for } p \in V_0, \\
\mathcal{M}, s \models \neg \alpha & \quad \text{iff} \quad \mathcal{M}, s \not\models \alpha, \\
\mathcal{M}, s \models \alpha \vee \beta & \quad \text{iff} \quad \mathcal{M}, s \models \alpha \text{ or } \mathcal{M}, s \models \beta, \\
\mathcal{M}, s \models M_i^d \alpha & \quad \text{iff} \quad |\{s' \in S : (s, s') \in RB(i) \text{ and } \mathcal{M}, s' \models \alpha\}| > d, d \in \mathbb{N}, \\
\mathcal{M}, s \models \diamond(i : P) \alpha & \quad \text{iff} \quad \exists s' \in S ((s, s') \in I_\Pi(P)(i) \text{ and } \mathcal{M}, s' \models \alpha).
\end{aligned}$$

We say that  $\alpha$  is *true* in a model  $\mathcal{M}$  at a state  $s$  if  $\mathcal{M}, s \models \alpha$ . Formula  $\alpha$  is *true* in  $\mathcal{M}$  ( $\mathcal{M} \models \alpha$ ) if  $\mathcal{M}, s \models \alpha$  for all  $s \in S$ , and  $\alpha$  is called *valid* ( $\models \alpha$ ) if  $\mathcal{M} \models \alpha$  for all  $\mathcal{M}$ .

## 2.2. Axiomatization

In this subsection we characterize the semantic consequence operation described above in syntactic terms and thereby we give a formal deductive system for deducing properties of persuasion process expressible in the language of  $\mathcal{AG}_n$ .

**Definition 2.3.** The system  $\mathcal{AG}_n$  is defined as follows. It has three inference rules:

$$\mathbf{R1} \frac{\alpha, \alpha \rightarrow \beta}{\beta} \qquad \mathbf{R2} \frac{\alpha}{B_i^0 \alpha} \qquad \mathbf{R3} \frac{\alpha}{\Box(i:P)\alpha}$$

It has also the following axioms:

**A0** classical propositional tautologies

$$\mathbf{A1} M_i^{d+1} \alpha \rightarrow M_i^d \alpha$$

$$\mathbf{A2} B_i^0(\alpha \rightarrow \beta) \rightarrow (M_i^d \alpha \rightarrow M_i^d \beta)$$

$$\mathbf{A3} M_i^0(\alpha \wedge \beta) \rightarrow ((M_i^{d_1} \alpha \wedge M_i^{d_2} \beta) \rightarrow M_i^{d_1+d_2}(\alpha \vee \beta))$$

$$\mathbf{A4} M_i^d \alpha \rightarrow B_i^0 M_i^d \alpha$$

$$\mathbf{A5} M_i^0 M_i^d \alpha \rightarrow M_i^d \alpha$$

$$\mathbf{A6} M_i^0(\text{true})$$

$$\mathbf{A7} \Box(i:P)(\alpha \rightarrow \beta) \rightarrow (\Box(i:P)\alpha \rightarrow \Box(i:P)\beta)$$

$$\mathbf{A8} \Box(i:P)(\alpha \wedge \beta) \leftrightarrow (\Box(i:P)\alpha \wedge \Box(i:P)\beta)$$

$$\mathbf{A9} \Box(i:P_1;P_2)\alpha \leftrightarrow \Box(i:P_1)(\Box(i:P_2)\alpha)$$

$$\mathbf{A10} \Box(i:P)\alpha \rightarrow \Diamond(i:P)\alpha$$

$$\mathbf{A11} \Box(i:P)\text{true}$$

$$\mathbf{A12} \Box(i:Id)\alpha \leftrightarrow \alpha$$

In all the above schemes of axioms, the symbols  $P, P_1, P_2$  denote schemes of programs,  $d, d_1, d_2$  are natural numbers,  $\alpha, \beta$  are arbitrary formulas and  $i$  is a name of an agent,  $i \in \text{Agt}$ .

The rules **R1** (Modus Ponens), **R2** (Necessitation for graded beliefs) and axioms **A0-A4** are equivalents of rules and axioms of Logic of Graded Modalities (LGM) which was introduced in papers by M. Fattorosi-Barnaba, F. de Caro, and C. Cerrato [7, 11, 12]. Later, its epistemic interpretation was given by W. van der Hoek and J.-J. Ch. Meyer [15]. The main differences between our approach and the one proposed by those authors are: first, we have a logic for many agents (not only one as they assume) and second, we explore a doxastic model rather than the epistemic one (as in Hoek-Meyer version). Thereby we assume that doxastic accessibility relations are serial, transitive, and Euclidean instead of equivalence relations. As a consequence the axiomatic system is a bit changed when compared to the original one of Fattorosi-Barnaba, de Caro and Cerrato as well as when compared to the epistemic version of Hoek-Meyer. More specifically, we use the logic of de Caro as a basis for syntactic formulation of the axioms

and the logic of Hoek-Meyer as a basis for our semantics. However, still their proposals are modified since we intend to capture the attributes of beliefs instead of knowledge (or in terms of axiomatic system - we are interested in the weak- $S5_n$  system not in the  $S5_n$  one). Axioms **A1**, **A2**, **A3** correspond to the axiom K of commonly used (non-graded)  $S5_n$  or weak- $S5_n$  modal system. For more details see [7]. In the paper of de Caro and the one of van der Hoek and Meyer, syntax of formulas that create their axiom systems (especially axiom **A3**) are different but it is easy to verify that both systems are equivalent. Axioms **A4**, **A5** and **A6** correspond to the axioms 5, 4, and D in weak- $S5_n$  system, respectively. They hold in models for which accessibility relation is Euclidean, transitive and serial, respectively.

The rule **R3** (Necessitation for programs) and axioms **A7-A12** find their motivation in the same fashion as the corresponding rules and axioms in Algorithmic Logic (AL) (cf. [19]) and Dynamic Logic (DL) (cf. [14]). However, in AL and DL it is not considered who is a performer of a given program. Therefore, axioms of  $\mathcal{AG}_n$  are similar but not exactly the same. As we noted, in AL and DL there are far more program constructions which we do not need in this approach.

We write  $\mathcal{AG}_n \vdash \alpha$  if the formula  $\alpha$  is provable in the deductive system. Moreover, we say that a formula  $\alpha$  is *consistent* if  $\mathcal{AG}_n \not\vdash \neg\alpha$ , that is, if it is not the case that  $\mathcal{AG}_n \vdash \neg\alpha$ . A finite set  $\{\phi_1, \dots, \phi_k\}$  of formulas is called *consistent* if its conjunction  $\phi_1 \wedge \dots \wedge \phi_k$  is consistent. An infinite set of formulas is called *consistent* if every finite subset is consistent. Formulas and sets of formulas are called *inconsistent* if they are not consistent. A set  $\Gamma$  of formulas is *maximally consistent* if  $\Gamma$  is consistent and  $\Gamma \cup \{\psi\}$  is inconsistent for any formula  $\psi \notin \Gamma$ .

### 3. Soundness and completeness

In this section we show that the deductive system  $\mathcal{AG}_n$  is sound and complete, i.e. that all theorems are valid formulas and all valid formulas are theorems. To prove this fact, we use the well known technique of the canonical models by Lemmon and Scott for classical modal logics. More precisely, we apply the Henkin's method. That is, we define a satisfying model for any maximally consistent set of formulas  $\Gamma$  such that its frame is a frame for  $\Gamma$ .

The proof is based on the completeness results for normal logics with graded modalities (NLGM-s) (see [7, 12, 11]), epistemic logics (see [10]) and dynamic logics (see [14]). Some definitions and theorems of this section are quoted from the above works. However, they are modified for the needs of this paper.

#### Theorem 3.1. (soundness)

$\mathcal{AG}_n$  is sound with respect to  $\mathcal{M}$ .

Using the standard manner, it is easy to show that  $\mathcal{M}$  satisfies the axioms of the system  $\mathcal{AG}_n$  and the rules of the system  $\mathcal{AG}_n$ . The rules hold in the sense that, if their premises are valid, then the consequents are valid as well (c.f. [11, 15, 18, 14]). An inductive proof on the length of derivations then yields that every provable formula is true. Below we justify that the rule **R2** preserves validity and the axioms **A1-A6** are valid formulas. The proof for the rule **R3** and the axioms **A7-A12** is analogous.

**R2** Assume that for any model  $\mathcal{M}$  and any state  $s$  of this model  $\mathcal{M}$ ,  $s \models \alpha$  holds and there exist a model  $\mathcal{M}'$  and a state  $s'$  of this model such that  $\mathcal{M}', s' \not\models B_i^0 \alpha$ . Then (by definition of the operator  $B$ ),  $\mathcal{M}', s' \models M_i^0 \neg\alpha$ , i.e. (by the definition of relation  $\models$ ) there exists a state  $s''$  such that  $(s', s'') \in RB(i)$  and  $\mathcal{M}', s'' \models \neg\alpha$ , what contradicts the initial assumption.



**A1** Let  $\mathcal{M}$  be a model and  $s$  be a state of this model. Assume that  $\mathcal{M}, s \models M_i^{d+1}\alpha$ . Then,  $|\{s' \in S : (s, s') \in RB(i) \text{ and } \mathcal{M}, s' \models \alpha\}| > d + 1$ . Hence,  $|\{s' \in S : (s, s') \in RB(i) \text{ and } \mathcal{M}, s' \models \alpha\}| > d$ , i.e.  $\mathcal{M}, s \models M_i^d\alpha$ .

**A2** Let  $\mathcal{M}$  be a model and  $s$  be a state of this model. Assume that  $\mathcal{M}, s \models B_i^0(\alpha \rightarrow \beta)$ . Then,  $|\{s' \in S : (s, s') \in RB(i) \text{ and } \mathcal{M}, s' \models \neg(\alpha \rightarrow \beta)\}| = 0$ . Therefore, for any state  $s'$ , if  $(s, s') \in RB(i)$  and  $\mathcal{M}, s' \models \alpha$ , then  $\mathcal{M}, s' \models \beta$ . Next, assume that  $\mathcal{M}, s \models M_i^d\alpha$ , i.e., there are more than  $d$  states  $s'$  such that  $(s, s') \in RB(i)$  and  $\mathcal{M}, s' \models \alpha$ . Thereby, there are more than  $d$  states  $s'$  such that  $(s, s') \in RB(i)$  and  $\mathcal{M}, s' \models \beta$ . Hence,  $|\{s' \in S : (s, s') \in RB(i) \text{ and } \mathcal{M}, s' \models \beta\}| > d$ , i.e.  $\mathcal{M}, s \models M_i^d\beta$ .

**A3** Let  $\mathcal{M}$  be a model and  $s$  be a state of this model. Assume that  $\mathcal{M}, s \models M_i^0(\alpha \wedge \beta)$ , i.e. there are no states  $s'$  such that  $(s, s') \in RB(i)$  and  $\mathcal{M}, s' \models \alpha \wedge \beta$ . Next assume that  $\mathcal{M}, s \models M_i^{d_1}\alpha$  and  $\mathcal{M}, s \models M_i^{d_2}\beta$ . Thus, there are exactly  $d_1$  states which are in relation  $RB(i)$  with the state  $s$  and satisfy  $\alpha$  and there are exactly  $d_2$  states which are in relation  $RB(i)$  with the state  $s$  and satisfy  $\beta$ . Moreover, there are no states in relation  $RB(i)$  with the state  $s$ , in which both  $\alpha$  and  $\beta$  are true. Therefore there are exactly  $d_1 + d_2$  states  $s'$  such that  $(s, s') \in RB(i)$  and  $\mathcal{M}, s' \models \alpha \vee \beta$ , i.e.,  $\mathcal{M}, s \models M_i^{d_1+d_2}(\alpha \vee \beta)$ .

**A4** Let  $\mathcal{M}$  be a model and  $s$  be a state of this model. Assume that  $\mathcal{M}, s \models M_i^d\alpha$ , i.e. there are more than  $d$  states such that  $(s, s') \in RB(i)$  and  $\mathcal{M}, s' \models \alpha$ . Let  $s_0$  be such a state. Since  $RB(i)$  is Euclidean  $s_0$  is in relation  $RB(i)$  with more than  $d$  states  $s''$  such that  $\mathcal{M}, s'' \models \alpha$ . Therefore,  $\mathcal{M}, s_0 \models M_i^d\alpha$ . Thus, for any state  $s'$  such that  $(s, s') \in RB(i)$  and  $\mathcal{M}, s' \models \alpha$  it holds that  $\mathcal{M}, s' \models M_i^d\alpha$ , i.e.,  $\mathcal{M}, s \models B_i^0M_i^d\alpha$ .

**A5** Let  $\mathcal{M}$  be a model and  $s$  be a state of this model. Assume that  $\mathcal{M}, s \models M_i^0M_i^d\alpha$ . Then,  $|\{s' \in S : (s, s') \in RB(i) \text{ and } \mathcal{M}, s' \models M_i^d\alpha\}| > 0$ . Hence, there exists at least one state  $s'$  such that there are more than  $d$  states  $s''$  such that  $(s', s'') \in RB(i)$  and  $\mathcal{M}, s'' \models \alpha$ . Since the relation  $RB(i)$  is transitive there are more than  $d$  states  $s''$  such that  $(s, s'') \in RB(i)$  and  $\mathcal{M}, s'' \models \alpha$ . Therefore,  $\mathcal{M}, s \models M_i^d\alpha$ .

**A6** Let  $\mathcal{M}$  be a model and  $s$  be a state of this model. Since the relation  $RB(i)$  is serial there exists at least one state  $s'$  such that  $(s, s') \in RB(i)$ . Thus, there exists at least one state  $s'$  such that  $(s, s') \in RB(i)$  and  $\mathcal{M}, s' \models true$ , i.e.,  $\mathcal{M}, s \models M_i^0(true)$ .

### Theorem 3.2. (completeness)

$\mathcal{AG}_n$  is complete with respect to  $\mathcal{M}$ .

To prove this theorem we have to show that, for all  $\alpha \in F$ , if  $\models \alpha$ , then  $\mathcal{AG}_n \vdash \alpha$  or equivalently, if  $\mathcal{AG}_n \not\vdash \alpha$ , then  $\not\models \alpha$ , i.e. if  $\mathcal{AG}_n \not\vdash \alpha$ , then there exists a model  $\mathcal{M}$  and a state  $s$  of this model such that  $\mathcal{M}, s \models \neg\alpha$ . Thus, (by replacing  $\alpha$  by  $\neg\alpha$ ) if  $\mathcal{AG}_n \not\vdash \neg\alpha$ , then there exists a model  $\mathcal{M}$  and a state  $s$  of this model such that  $\mathcal{M}, s \models \alpha$ . Thus, proving completeness is equivalent to showing that every consistent formula is satisfiable. In order to prove this, it is sufficient to show that every consistent set of formulas is satisfiable. Since every consistent set of formulas can be extended to a maximally consistent set, it is sufficient to show that every maximally consistent set of formulas is satisfiable. This is established by means of the construction of *canonical* Kripke structure  $\mathcal{M}^c$ .

Let  $\Phi$  be the set of all the maximally consistent sets.

**Definition 3.1.** The function  $m_i : \Phi \times \Phi \rightarrow \omega + 1 = \omega \cup \{\omega\}$ , for  $i \in Agt$  is defined as follows: for every  $\Gamma, \Gamma' \in \Phi$

- $m_i(\Gamma, \Gamma') = \omega$  if for any  $\alpha \in \Gamma'$ ,  $M_i^d \alpha \in \Gamma$  for any  $d \in \mathbb{N}$ ,
- $m_i(\Gamma, \Gamma') = h = \min\{d \in \mathbb{N} : M_i^d \alpha \in \Gamma \text{ and } \alpha \in \Gamma'\}$  otherwise.

**Definition 3.2.** Let  $\Gamma_0 \in \Phi$ . The set

$$SF_i(\Gamma_0) = \bigcup \{ \{\Gamma\} \times m_i(\Gamma_0, \Gamma) : \Gamma \in \Phi \}$$

for  $i \in \text{Agt}$ , will be called the satisfying family of  $\Gamma_0$ .

An element of  $SF_i(\Gamma_0)$  is of the form  $\langle \Gamma, d \rangle$  where  $d < m_i(\Gamma_0, \Gamma)$ . Therefore we shall think of  $SF_i(\Gamma_0)$  as it is made up by  $m_i(\Gamma_0, \Gamma)$  ordered copies of  $\Gamma$ , for any  $\Gamma \in \Phi$ .

**Theorem 3.3.** For any  $\alpha \in F$ ,  $i \in \text{Agt}$ , and  $d \in \mathbb{N}$ ,

$$M_i^d \alpha \in \Gamma_0 \text{ iff } |\{\Gamma \in SF_i(\Gamma_0) : \alpha \in \Gamma\}| > d$$

where, to simplify notations, we identify a couple  $\langle \Gamma, d \rangle$  ( $d < m(\Gamma_0, \Gamma)$ ) with its first component.

For the proof see [7].

Let

$$m(\Gamma) = \sup\{m_i(\Gamma', \Gamma) : \Gamma' \in \Phi \text{ and } i \in \text{Agt}\}$$

for any  $\Gamma \in \Phi$ .

The canonical Kripke model we define as

$$\mathcal{M}^c = (S^c, RB^c, I^c, v^c)$$

with

- $S^c = \bigcup \{ \{\Gamma\} \times m(\Gamma) : \Gamma \in \Phi \} \cup \Psi$ , where  $\Psi = \{ \langle \Gamma, \omega \rangle : \Gamma \in \Phi \text{ and } m(\Gamma) = 0 \}$ . We may think of  $S^c$  as made up by  $m(\Gamma)$  ordered copies of  $\Gamma$ , if  $m(\Gamma) \neq 0$ , and by one copy of  $\Gamma$ , if  $m(\Gamma) = 0$ , for any  $\Gamma \in \Phi$ . We shall identify  $\langle \Gamma, d \rangle$  ( $d \leq \omega$ ) with  $s_\Gamma$ .
- $RB^c : \text{Agt} \longrightarrow 2^{S \times S}$  is a function such that  $RB^c(i) = \{ (s_\Gamma, s_{\Gamma'}) \in S \times S : s_{\Gamma'} \in SF_i(\Gamma) \}$ ,
- $I^c : \Pi_0 \longrightarrow (\text{Agt} \longrightarrow 2^{S \times S})$  is a function such that  $I^c(a)(i) = \{ (s_\Gamma, s_{\Gamma'}) : \forall \alpha \in F \text{ (if } \Box(i : a)\alpha \in \Gamma, \text{ then } \alpha \in \Gamma') \}$ ,
- $v^c : S \longrightarrow \{ \mathbf{0}, \mathbf{1} \}^{V_0}$  is a function such that  $v^c(s_\Gamma)(p) = \mathbf{1}$  iff  $p \in \Gamma$ .

First, we prove by induction on the complexity of program scheme  $P$  that  $I_\Pi^c : \Pi \longrightarrow (\text{Agt} \longrightarrow 2^{S \times S})$  is a function such that  $I_\Pi^c(P)(i) = \{ (s_\Gamma, s_{\Gamma'}) : \forall \alpha \in F \text{ (if } \Box(i : P)\alpha \in \Gamma, \text{ then } \alpha \in \Gamma') \}$ . If  $P = a \in \Pi_0$ , then the thesis follows from the definition of the canonical model. Suppose that  $P = P_1; P_2$  and  $(s_\Gamma, s_{\Gamma'}) \in I_\Pi^c(P)(i)$ .

( $\Rightarrow$ ) Let  $s_\Gamma, s_{\Gamma'}$  be states such that  $(s_\Gamma, s_{\Gamma'}) \in I_\Pi^c(P_1; P_2)(i)$ . Then, there exists a state  $s_{\Gamma''}$  such that  $(s_\Gamma, s_{\Gamma''}) \in I_\Pi^c(P_1)(i)$  and  $(s_{\Gamma''}, s_{\Gamma'}) \in I_\Pi^c(P_2)(i)$ . Suppose that  $\Box(i : P_1; P_2)\alpha \in \Gamma$ . Then, by axiom A9 and maximal consistency of  $\Gamma$ ,  $\Box(i : P_1)\Box(i : P_2)\alpha \in \Gamma$ . Now, by inductive hypothesis,  $\Box(i : P_2)\alpha \in \Gamma''$  and  $\alpha \in \Gamma'$ .

( $\Leftarrow$ ) *Claim.* Let  $\Gamma$  be a maximally consistent set of formulas. Then, there exists a maximally consistent set  $\Gamma'$  such that  $\alpha \in \Gamma'$  for any formula  $\alpha$  such that  $\Box(i : P)\alpha \in \Gamma$ .

**Proof of the Claim.** Let  $\Gamma(P)(i) = \{\alpha : \Box(i : P)\alpha \in \Gamma\}$ . Observe that  $\Gamma(P)(i)$  is a non-empty set since, by axiom A11,  $true \in \Gamma(P)(i)$ . We shall prove that  $\Gamma(P)(i)$  is a consistent set.

Suppose  $\Gamma(P)(i)$  is inconsistent. Then there exists a finite set  $\{\phi_1, \dots, \phi_k\} \subseteq \Gamma(P)(i)$  such that  $\mathcal{AG}_n \vdash \neg(\phi_1 \wedge \dots \wedge \phi_k)$ . Now, by R3,  $\mathcal{AG}_n \vdash \Box(i : P)(\neg(\phi_1 \wedge \dots \wedge \phi_k))$  and, by A10,  $\mathcal{AG}_n \vdash \Diamond(i : P)(\neg(\phi_1 \wedge \dots \wedge \phi_k))$ , i.e. by the definition of operator  $\Box$ ,  $\mathcal{AG}_n \vdash \neg\Box(i : P)(\phi_1 \wedge \dots \wedge \phi_k)$ . Thus, by A8,  $\mathcal{AG}_n \vdash \neg(\Box(i : P)\phi_1 \wedge \dots \wedge \Box(i : P)\phi_k)$ . As a consequence, the set  $\{\Box(i : P)\phi_j : j = 1, \dots, k\}$  is inconsistent. This contradicts the assumption that  $\{\Box(i : P)\phi_j : j = 1, \dots, k\} \subseteq \Gamma$  and  $\Gamma$  is a maximally consistent set. Therefore,  $\Gamma(P)(i)$  is consistent.

Since any consistent set can be extended to a maximally consistent set there exists a maximally consistent set  $\Gamma'$  such that  $\Gamma(P)(i) \subseteq \Gamma'$ , what ends the proof of the Claim.

Let  $\Gamma, \Gamma'$  be maximally consistent sets. Assume that if  $\Box(i : P_1; P_2)\alpha \in \Gamma$ , then  $\alpha \in \Gamma'$  for any  $\alpha \in F$ . Then, by axiom A9 and maximal consistency of  $\Gamma$ ,  $\Box(i : P_1)\Box(i : P_2)\alpha \in \Gamma$ . Now, by the Claim, there exists a maximally consistent set  $\Gamma''$  such that  $\Box(i : P_2)\alpha \in \Gamma''$  and, by inductive hypothesis,  $(s_\Gamma, s_{\Gamma''}) \in I_{\Pi}^c(P_1)(i)$  and  $(s_{\Gamma''}, s_{\Gamma'}) \in I_{\Pi}^c(P_2)(i)$ . Thus,  $(s_\Gamma, s_{\Gamma'}) \in I_{\Pi}^c(P_1; P_2)(i)$ .

**Lemma 3.1.** For any maximally consistent  $\Gamma$ , it holds that for any  $\alpha$ ,  $M^c, s_\Gamma \models \alpha$  iff  $\alpha \in \Gamma$ .

**Proof** The proof by induction on the structure of  $\alpha$ .

**Case 1**  $\alpha = p \in V_0$ . Directly from the definition of  $v^c$ .

**Case 2**  $\alpha = \beta_1 \vee \beta_2$ .

$M^c, s_\Gamma \models \beta_1 \vee \beta_2$  iff  $M^c, s_\Gamma \models \beta_1$  or  $M^c, s_\Gamma \models \beta_2$  iff (by the inductive hypothesis)  $\beta_1 \in \Gamma$  or  $\beta_2 \in \Gamma$  iff (by maximal consistency of  $\Gamma$ )  $\beta_1 \vee \beta_2 \in \Gamma$ .

**Case 3**  $\alpha = \neg\beta$ .

$M^c, s_\Gamma \models \neg\beta$  iff  $M^c, s_\Gamma \not\models \beta$  iff (by the inductive hypothesis)  $\beta \notin \Gamma$  iff (by maximal consistency of  $\Gamma$ )  $\neg\beta \in \Gamma$ .

**Case 4**  $\alpha = \Box(i : P)\beta$ .

( $\Leftarrow$ ) Suppose  $\Box(i : P)\beta \in \Gamma$ . Then, by definition of  $I_{\Pi}^c$ , if  $(s_\Gamma, s_{\Gamma'}) \in I_{\Pi}^c(P)(i)$ , then  $\beta \in \Gamma'$ . Thus using the inductive hypothesis,  $M^c, s_{\Gamma'} \models \beta$  for all  $\Gamma'$  such that  $(s_\Gamma, s_{\Gamma'}) \in I_{\Pi}^c(P)(i)$ , i.e.  $M^c, s_\Gamma \models \Box(i : P)\beta$ .

( $\Rightarrow$ ) For the other direction, assume  $M^c, s_\Gamma \models \Box(i : P)\beta$ . It is easy to show that the set  $\{\phi : \Box(i : P)\phi \in \Gamma\} \cup \{\neg\beta\}$  is inconsistent. From this it follows that there must be some finite subset, say  $\{\phi_1, \dots, \phi_k, \neg\beta\}$ , which is inconsistent. Thus, by propositional reasoning, we have

$$\mathcal{AG}_n \vdash \phi_1 \rightarrow (\phi_2 \rightarrow (\dots \rightarrow (\phi_k \rightarrow \beta) \dots)).$$

By R3, we have

$$\mathcal{AG}_n \vdash \Box(i : P)(\phi_1 \rightarrow (\phi_2 \rightarrow (\dots \rightarrow (\phi_k \rightarrow \beta) \dots))).$$

By induction on  $k$ , together with axiom A7 and propositional reasoning, we can show

$$\begin{aligned} & \mathcal{AG}_n \vdash \Box(i : P)(\phi_1 \rightarrow (\phi_2 \rightarrow (\dots \rightarrow (\phi_k \rightarrow \beta) \dots))) \\ & \rightarrow (\Box(i : P)\phi_1 \rightarrow (\Box(i : P)\phi_2 \rightarrow (\dots \rightarrow (\Box(i : P)\phi_k \rightarrow \Box(i : P)\beta) \dots))). \end{aligned}$$

Now from R1, we get

$$\mathcal{AG}_n \vdash \Box(i : P)\phi_1 \rightarrow (\Box(i : P)\phi_2 \rightarrow (\dots \rightarrow (\Box(i : P)\phi_k \rightarrow \Box(i : P)\beta) \dots)).$$

Hence (since  $\Gamma$  contains all theorems)

$$\Box(i : P)\phi_1 \rightarrow (\Box(i : P)\phi_2 \rightarrow (\dots \rightarrow (\Box(i : P)\phi_k \rightarrow \Box(i : P)\beta) \dots)) \in \Gamma.$$

Because  $\phi_1, \dots, \phi_k \in \{\phi : \Box(i : P)\phi \in \Gamma\}$ , we must have  $\Box(i : P)\phi_1, \dots, \Box(i : P)\phi_k \in \Gamma$ . Now, from the maximal consistency of  $\Gamma$ , it follows that  $\Box(i : P)\beta \in \Gamma$ .

**Case 5**  $\alpha = M_i^d \beta$ .

$\mathcal{M}^c, s_\Gamma \models M_i^d \beta$  iff (by the inductive hypothesis)  $|\{s_{\Gamma'} : (s_\Gamma, s_{\Gamma'}) \in RB^c(i) \text{ and } \beta \in \Gamma'\}| > d$  iff (by the definition of the canonical structure)  $|\{s_{\Gamma'} : s_{\Gamma'} \in SF_i(\Gamma) \text{ and } \beta \in \Gamma'\}| > d$  iff (by Theorem 3.3)  $M_i^d \beta \in \Gamma$ .

Notice that, Lemma 3.1 implies  $\mathcal{M}^c, s_\Gamma \models \Gamma$ , for any maximally consistent set  $\Gamma$ . Thereby, we showed that every consistent set  $\Gamma$  is satisfiable what ends the completeness proof.

## 4. Examples of expressiveness

Our research aim is to reason about a persuasion process performed in a multi-agent system. To this end we need to employ a formal language in which it is possible to characterize main attributes of this process. In this section we show that  $\mathcal{AG}_n$  logic is expressible enough to accomplish this task. Among the important features of argumentation are: existence of a conflict of opinion, success of persuasion, power of arguments as well as an influence of a proponent credibility on the effect of an argumentation.

Consider two agents which are committed to make an attempt to cause the rise of the environment's temperature as soon as it falls under 10<sup>0</sup>C. Assume that one agent is not able to carry out the task on his own. It is possible only if agents cooperate.

**Example 1 (Conflict of opinion)** The condition for persuasion to start is a conflict of opinion. In our scenario a conflict appears when one of the agents believes that the temperature is lower than 10<sup>0</sup>C while the other does not - possibly because agents use different sources of information and thereby derive different conclusions. Indeed, as long as agents have the same opinion, there is no use to start the persuasion. Only a conflict might make one of agents try to convince the other to his ideas. Recall that an agent which gives a thesis and defends it is a proponent while an agent to which the persuasion is addressed is an audience.<sup>5</sup> Interestingly, a conflict appears not only when a proponent is absolutely sure about the thesis and an audience is absolutely against it. It can also arise from the fact that the degrees of agents' beliefs differ or belong to different intervals, where by a degree we understand the ratio of the number of states which the agent considers and at which a thesis is true to the number of all states he considers as his doxastic alternatives. Say that degrees from  $(\frac{1}{2}, 1]$  mean accepting the thesis and degrees from  $[0, \frac{1}{2}]$  mean rejecting the thesis. Then, the conflict of opinion can be captured in the language of  $\mathcal{AG}_n$  in the following way:

$$M_{prop}^{!3,4}(p_{t < 10}) \wedge M_{aud}^{!1,4}(p_{t < 10})$$

<sup>5</sup>In our example, the second agent disagrees and is persuaded at the same time. This means that he plays both the role of opponent and audience. However, for the clarity of the examples we will call him just "the audience".

where *prop* and *aud* mean the proponent and the audience, respectively, and  $p_{t < 10}$  is a propositional variable which expresses that the temperature is lower than  $10^0\text{C}$ . This formula should be read as follows: “The proponent believes that the temperature is lower than  $10^0\text{C}$  with the degree  $\frac{3}{4}$  and the audience believes the temperature is lower than  $10^0\text{C}$  with the degree  $\frac{1}{4}$ ”.

**Example 2 (Success of persuasion)** The persuasion  $P = (a_1; a_2; \dots; a_k)$  can be successful when after performing arguments  $a_1, a_2, \dots, a_k$  by the proponent, it is possible that the audience will believe the thesis with some expected degree. Recall that in our scenario an agent accepts the thesis if he believes it in a degree higher than  $\frac{1}{2}$ . Then, the *success* will be achieved when:

$$\diamond(prop : P)(M_{aud}^{3,4}(p_{t < 10})).$$

This formula states “If the proponent performs arguments  $P$  then it is possible that the audience will believe the thesis with the degree  $\frac{3}{4}$ ”.

Interestingly, success can be also *subjective*. That is, after execution of  $P$  the proponent may believe that he achieved a goal while he did not. This means that he faulty evaluated the results of his persuasion:

$$\diamond(prop : P)[M_{prop}^{4,4}(M_{aud}^{3,4}(p_{t < 10})) \wedge \neg M_{aud}^{3,4}(p_{t < 10})].$$

The formula states “If the proponent performs arguments  $P$  then it is possible he will believe that the audience is convinced with the degree  $\frac{3}{4}$ , but the audience will not believe the thesis with this degree”.

The other important issue is whether the proponent believes (*predicts*) that he is able to succeed. Otherwise, he may not start convincing even though he had all necessary means to prevail in the desired degree. Such a situation can be expressed by the formula:

$$M_{prop}^{0,4}[\diamond(prop : P)(M_{aud}^{4,4}(p_{t < 10}))] \wedge \square(prop : P)(M_{aud}^{4,4}(p_{t < 10})).$$

Here the proponent is absolutely sure that his persuasion  $P$  will fail (the proponent believes with the degree  $\frac{0}{4}$  that the audience may become convinced to the thesis with the degree  $\frac{4}{4}$ ), but in fact  $P$  would lead him to success (after persuasion  $P$  the audience will believe the thesis with the desired degree  $\frac{4}{4}$ ).

**Example 3 (Power of arguments)** The persuasiveness may depend on the quality of arguments. Assuming the same proponent and the same audience but different arguments, we can obtain results of unlike strength. Say that if in the scenario the proponent gives verbal argument “One of your thermometers is placed wrongly since it is too close to a heater” (action  $a$ ), then he will prevail. However, his success will not be absolute:

$$M_{aud}^{1,4}(p_{t < 10}) \rightarrow \square(prop : a)(M_{aud}^{3,4}(p_{t < 10})).$$

We read this formula as follows “If the audience believes the thesis with the degree  $\frac{1}{4}$  then always after the execution of the action  $a$  by the proponent, the audience will believe the thesis with the degree  $\frac{3}{4}$ ”.

When in the same situation the proponent moves the thermometer to another place (action  $b$ ) and thus proves that the temperature is lower than  $10^0\text{C}$ , he may obtain audience’s utter conviction:

$$M_{aud}^{1,4}(p_{t < 10}) \rightarrow \diamond(prop : b)(M_{aud}^{4,4}(p_{t < 10})).$$

**Example 4 (Power of proponent’s credibility)** Another important factor which affects the persuasiveness is a proponent, precisely - who he is. Assume that the audience finds a proponent (call him

$prop_1$ ) unreliable. As a consequence, he trusts nothing what the proponent says or acts and therefore none of his arguments will convince him. On the other hand, if another proponent (call him  $prop_2$ ) is a leader of a group of agents or is a specialist, then his arguments have great persuasive power. Simply stated, the same arguments can cause different results depending on an agent who performs them:

$$\neg\Diamond(prop_1 : P)(M!_{aud}^{4,4}(p_{t < 10})) \wedge \Diamond(prop_2 : P)(M!_{aud}^{4,4}(p_{t < 10})).$$

## 5. Conclusions

In the article, we introduce the **deductive system**  $\mathcal{AG}_n$ , i.e., we define its syntax and semantics as well as provide an elegant axiomatization for the resulting language. We also show the *soundness* and *completeness* theorems for the logic. Having the outcome of completeness, we can apply  $\mathcal{AG}_n$  for specification and verification of properties of multi-agent systems in which agents perform the process of convincing. In particular, the complete logic allows us to employ it in *axiomatic* verification. This means that our logic provides a formal proof of correctness of distributed systems and creates the possibility of finding faults. Moreover, we show how **expressible** the language of our logic is with respect to the persuasion process. With the help of  $\mathcal{AG}_n$ , we can reason about the effects which convincing brings about - in what degree the persuasion changes audience's beliefs, what tactic should an agent undertake to influence other agent's opinion, whether the persuasion was successful, how strong the victory is, whether the success is real or only subjective, what agent has sufficient persuasive power to convince a given audience, etc. On the basis of  $\mathcal{AG}_n$  logic we plan to develop a software system **investigating** a persuasion process. The aim of this system will be to study multi-agent systems in which agents can argue and perform actions to influence each others beliefs.

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